Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 5, 2003*
Accepted: *September 12, 2003*

Strong interactions and stability in the DGP model

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Abstract: The model of Dvali, Gabadadze, and Porrati (DGP) gives a simple geometrical setup in which gravity becomes 5-dimensional at distances larger than a length scale $\lambda_{\rm DGP}$. We show that this theory has strong interactions at a length scale $\lambda_3 \sim (\lambda_{\rm DGP}^2/M_{\rm P})^{1/3}$. If $\lambda_{\rm DGP}$ is of order the Hubble length, then the theory loses predictivity at distances shorter than $\lambda_3 \sim 1000\,{\rm km}$. The strong interaction can be viewed as arising from a longitudinal 'eaten Goldstone' mode that gets a small kinetic term only from mixing with transverse graviton polarizations, analogous to the case of massive gravity. We also present a negative-energy classical solution, which can be avoided by cutting off the theory at the same scale scale λ_3. Finally, we examine the dynamics of the longitudinal Goldstone mode when the background geometry is curved.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

The DGP model [1] is the first ghost-free example of a mechanism in which gravity can be localized on a 4D brane in a space of infinite transverse volume. It describes a theory where 4D general covariance is unbroken, yet the graviton is a metastable state. Its main property is that, on the 4D brane, gravity looks 4D at short distance, while it weakens at large distance. This property suggest an interesting alternative to the standard description of our present-day accelerating universe. In DGP, the cosmic acceleration could be due to a modification of gravity at large (horizon-size) distance, rather than to a positive cosmological constant. Explicit realizations of this scenario have been proposed, for instance in ref. [2].

The model can be described by the action

$$S_{\rm DGP} = 2M_5^3 \int_{\mathcal{M}} d^5x\,\sqrt{-G}\,R(G) + \\ + \int_{\partial\mathcal{M}} d^4x\,\sqrt{-\gamma}\left[-4M_5^3 K(\gamma) + 2M_4^2 R(\gamma)\right], \tag{1.1}$$

where $\mathcal{M}$ is a 5D manifold with boundary $\partial\mathcal{M}$, G is the 5D metric, γ is the 4D induced metric on the boundary, and K is the extrinsic curvature.[1] In this model gravity on the brane looks 4D at distances shorter than

$$\lambda_{\rm DGP} = \frac{M_4^2}{M_5^3}\,. \tag{1.2}$$

For $M_4 \gg M_5$ this can be a macroscopic length, for example the size of the present horizon.

[1]The boundary can also be treated as an orbifold fixed point. We will discuss the relation between these approaches below.

The DGP model is closely related to massive gravity. In fact, the brane-to-brane graviton propagator in gaussian normal coordinates is

$$D^{\rm DGP}_{\mu\nu\,\rho\sigma}(p) = D^{\rm massive}_{\mu\nu\,\rho\sigma}(p, |p|/\lambda_{\rm DGP})\,, \tag{1.3}$$

where $D^{\rm massive}_{\mu\nu,\lambda\rho}(p, m^2)$ is the propagator for 4D massive gravity. The DGP therefore shares with massive gravity the 'vDVZ discontinuity' [3]: at distances smaller than $\lambda_{\rm DGP}$, the model reduces not to general relativity, but to a scalar-tensor theory of gravity, where the scalar couples with gravitational strength (it does not decouple in the limit $m \to 0$). Refs. [3] showed that in the one graviton exchange approximation, massive gravity predicts unacceptable deviations in the predicted bending of light by the sun.

However, as shown by Vainshtein [4] for massive gravity, the situation is actually more subtle. Near a heavy source, the one graviton exchange approximation breaks down at very large distances, and he argued that at smaller distances the resummation of nonlinear effects *restores* agreement with general relativity. In DGP, the one graviton exchange approximation breaks down at distances $R_* \sim (R_{\rm S}\lambda^2_{\rm DGP})^{1/3}$, where $R_{\rm S}$ is the Schwarzschild radius of the source. At smaller distances, it was explicitly shown that that the full nonlinear solution approaches that of general relativity [5, 6, 7]. Because the scale R_* is very large for astrophysical sources, it appeas that the DGP model may describe our universe [2].

The fact that the one-particle exchange breaks down at a distance so much larger than $R_{\rm S}$ suggests that DGP has hidden strong interaction scales. For a large classical source with $R_{\rm S} \gg 1/M_4$ the non-linearities at the scale R_* can certainly be associated to classical physics. On the other hand, for a source with $R_{\rm S} \sim 1/M_4$, corresponding heuristically to one quantum of gravitational charge, we expect any non-linearity to be due to quantum physics. Based on this qualitative argument we expect strong quantum effects to become important at a length scale

$$\lambda_3 = \left(\frac{\lambda^2_{\rm DGP}}{M_4}\right)^{1/3}. \tag{1.4}$$

In this paper we show that this is precisely what happens. For $\lambda_{\rm DGP}$ of order the Horizon size, $\lambda_3 \sim 1000\,{\rm km}$. At distances shorter than λ_3, new interactions become important, and there seems to be no reason that the theory should agree with general relativity.

An analogous strong interaction is also present in massive gravity. The strong interactions can be made manifest using the Stückelberg trick of nonlinearly realizing the gauge invariance broken by the mass term [8]. In massive gravity, the Stückelberg (or Goldstone) fields have strong self-interactions that necessitate a cutoff that goes to zero as the graviton mass goes to zero. An analogous phenomenon is familiar for massive non-abelian gauge fields, where the Stückelberg sector is a non-linear sigma model that is strongly interacting at a scale m/g, where m is the gauge boson mass and g is the gauge coupling. In the case of massive gravity, ref. [5] computed the off-shell three-graviton vertex and found a m^{-4} singularity, suggesting that the the theory should become strongly coupled at a scale $\Lambda_5 \sim (m^4 M_{\rm P})^{1/5}$. Indeed, ref. [8] explicitly showed that the theory does interact strongly at $\Lambda_5 \sim (m^4 M_{\rm P})^{1/5}$, or $\Lambda_3 \sim (m^2 M_{\rm P})^{1/3}$ if the leading strong terms are tuned to be small. Substituting the 'running mass' $|p|/\lambda_{\rm DGP}$ into the Λ_5 cutoff for massive gravity and solving for p also suggests that the DGP model has strong interactions at the scale eq. (1.4).

In this paper, we study the DGP model in detail to rigorously establish the existence of the strong interactions and understand their origin. Following the logic of the Stückelberg trick, we introduce extra pure gauge degrees of freedom to parameterize the strong interactions. We do this by formulating the theory on a space with boundary, with no a priori boundary conditions on the fields. This reduces to conventional orbifold boundary conditions in a particular gauge, but a different gauge choice is useful to make the strong interactions manifest. Since DGP is a generally covariant theory, it is not surprising that we find that the Stückelberg mode has a geometrical interpretation: it is a 'brane-bending' mode that keeps the induced boundary metric fixed.

We also find evidence for strong interactions at the scale eq. (1.4) at the classical level. We show that the DGP model has classical solutions with negative 5D energy, with a boundary stress tensor obeying the dominant energy condition. These solutions are at the edge of the regime of validity of the effective theory with short-distance cutoff given by eq. (1.4), giving another indication of new physics at that scale.

We then consider the behavior of the DGP model in the presence of curvature. We show that for the case of a positive curvature boundary (de Sitter sign), the self-interactions become stronger, and the Goldstone mode becomes a ghost for sufficiently large curvature. Closely related results have been found for massive gravity in refs. [9]. We also consider the Randall-Sundrum model [10] with a DGP kinetic term. For a DGP kinetic term on the Planck brane, we find no strong interactions, in agreement with expectations based on holography. For a DGP kinetic term on the IR brane, the radion becomes a ghost if $\lambda_{\rm DGP}$ becomes larger than the 5D AdS length.

This paper is organized as follows. Section 2 explains the boundary effective action we will use as a tool for the case of a toy scalar model. Section 3 uses this formalism to compute the boundary action for the DGP model. We find an effective action for the Stückelberg mode and explicitly compute the cubic interactions. We discuss the power counting and derive a non-renormalization theorem for the cubic interaction. Section 4 describes a negative energy solution. Section 5 extends the analysis of section 3 to the case of backgrounds with nonzero curvature.

2. Boundary effective action

In this section, we describe the formalism we use to obtain an effective action for the boundary field in a theory such as DGP. Consider a field theory on a space with boundary, and suppose that we are interested in the correlation functions of sources on the boundary. We do not impose any *a priori* boundary conditions on the fields. In the path integral, we integrate over arbitrary boundary values of the bulk fields weighted by their action.

It is useful to separate the fields into bulk fields Φ and boundary fields ϕ. Locality of the action means that the path integral can be written as

$$Z = \int d[\Phi]\, d[\phi]\, e^{i\left(S_{\rm bulk}[\Phi] + S_{\rm bdy}[\phi]\right)}. \tag{2.1}$$

Since the only sources are on the boundary we can integrate out the bulk fields to obtain a (nonlocal) effective action for the boundary fields. We must therefore integrate over all

Φ with boundary condition

$$\Phi| = \phi\,, \tag{2.2}$$

where '$|$' indicates evaluation at the boundary. We perform the Φ integral semi-classically, by expanding about a solution $\bar{\Phi}$ to the bulk equations of motion, with $\delta\Phi| = 0$ because of the boundary condition.[2] The semi-classical expression for the path integral is then

$$Z = \int d[\phi]\, e^{i\left(S_{\rm bdy}[\phi]+\Gamma[\phi]\right)}\,, \tag{2.3}$$

where the effective action from integrating out the bulk is

$$e^{i\Gamma[\phi]} = e^{iS_{\rm bulk}[\bar{\Phi}]} \int d[\Phi']\, \exp\left\{ \frac{i}{2} \int \Phi' \left. \frac{\delta^2 S_{\rm bulk}}{\delta\Phi^2} \right|_{\Phi=\bar{\Phi}} \Phi' + \cdots \right\}, \tag{2.4}$$

where the path integral over $\Phi' = \Phi - \bar{\Phi}$ is performed over fields with boundary condition $\Phi'| = 0$.

2.1 Scalar field theory

Let us do a simple example: free massless scalar field theory in a 5D space with 4D boundary at $y = 0$. The action is[3]

$$S = \int d^5x \left[-\frac{1}{2}\partial^M\Phi\partial_M\Phi \right] + \int d^4x \left[-\frac{1}{2}\kappa\partial^\mu\phi\partial_\mu\phi \right]_{y=0}, \tag{2.5}$$

where

$$\Phi| = \phi\,. \tag{2.6}$$

The classical solution for Φ with these boundary condition is

$$\bar{\Phi}(x,y) = e^{-y\Delta}\phi(x)\,, \tag{2.7}$$

where $\Delta = \sqrt{-\Box_4}$. We therefore obtain

$$\begin{aligned} \Gamma[\phi] &= \int d^5x \left[\frac{1}{2}\bar{\Phi}\Box_5\bar{\Phi} \right] + \int d^4x \left[\frac{1}{2}\bar{\Phi}\partial_y\bar{\Phi} \right]_{y=0} \\ &= -\int d^4x \left[\frac{1}{2}\phi\Delta\phi \right]. \end{aligned} \tag{2.8}$$

From this we can read off the propagator for the ϕ field:

$$\langle\phi\phi\rangle = \frac{1}{\kappa\Box_4 - \Delta}\,. \tag{2.9}$$

Now we add bulk interactions:

$$S_{\rm bulk} = \int d^5x \left[-\frac{1}{2}\partial^M\Phi\partial_M\phi - \frac{1}{3}\lambda\Phi^3 \right]. \tag{2.10}$$

[2]Because $\delta\Phi| = 0$ there is no boundary term in the variation of the bulk action.

[3]If we had written the bulk kinetic term as $\frac{1}{2}\Phi\Box_5\Phi$ there would be a boundary term in the variation proportional to $\Phi\partial_y(\delta\Phi)$, affecting the behavior of solutions near the boundary.

The classical bulk field satisfies

$$\Box_5 \bar{\Phi} - \lambda \bar{\Phi}^2 = 0\,, \qquad \bar{\Phi}| = \phi\,. \tag{2.11}$$

We find the solution order by order in λ:

$$\bar{\Phi} = \bar{\Phi}_0 + \bar{\Phi}_1 + \cdots\,, \tag{2.12}$$

where $\bar{\Phi}_n = \mathcal{O}(\lambda^n)$. $\bar{\Phi}_0$ was computed above. Because $\bar{\Phi}_0| = \phi$, we have

$$\bar{\Phi}_n| = 0 \qquad \text{for } n \geq 1\,. \tag{2.13}$$

We now compute the first-order correction to the action:

$$\Gamma[\phi] = \Gamma_0 + \Gamma_1 + \cdots, \tag{2.14}$$

where Γ_0 was computed above, and

$$\begin{aligned}
\Gamma_1 &= \int d^5x \left[-\partial^M \bar{\Phi}_1 \partial_M \bar{\Phi}_0 - \frac{1}{3}\lambda \bar{\Phi}_0^3 \right] \\
&= \int d^5x \left[\bar{\Phi}_1 \Box_5 \bar{\Phi}_0 - \frac{1}{3}\lambda \bar{\Phi}_0^3 \right] + \int d^4x \left[-\bar{\Phi}_1 \partial_y \bar{\Phi}_0 \right]_{y=0} \\
&= -\frac{1}{3}\lambda \int d^5x\, \bar{\Phi}_0^3 \\
&= -\frac{1}{3}\lambda \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \frac{\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)}{\sqrt{p_1^2} + \sqrt{p_2^2} + \sqrt{p_3^2}}\,,
\end{aligned} \tag{2.15}$$

where

$$\tilde{\phi}(p) = \int d^4x\, e^{ip\cdot x} \phi(x)\,. \tag{2.16}$$

3. Boundary effective action for gravity

We now consider 5D gravity with a 4D boundary, the case of interest. We use Latin capitals $M, N, \ldots = 0, \ldots, 3; 5$ for 5D spacetime indices, and $\mu, \nu, \ldots$ for 4D ones. We denote the bulk metric by G_{MN}.

In the standard treatment, we define boundary conditions by imposing an orbifold projection under reflections about the boundary. Here instead we will not impose any boundary conditions on G_{MN}. This means that there is extra gauge freedom in this formulation. In the bulk, we have infinitesimal gauge transformations generated by Ξ_M:

$$\delta G_{MN} = \Xi^P \partial_P G_{MN} + \partial_M \Xi^P G_{PN} + \partial_N \Xi^P G_{MP}\,. \tag{3.1}$$

In the orbifold formulation, we have $G_{5\mu}| = 0$ and hence $\dot{\Xi}_\mu| = 0$, $\Xi_5| = 0$, where the dot denotes the derivative with respect to x^5 and the vertical stroke denotes evaluation at the boundary. In the present formulation, $G_{5\mu}| \neq 0$ and $\dot{\Xi}_\mu| \neq 0$. We still have $\Xi^5| = 0$ because we use coordinates where the boundary position is fixed. We can choose a gauge where $G_{5\mu}| = 0$ to recover the orbifold boundary conditions, so this formulation is completely equivalent to the usual one. However, the extra gauge degrees of freedom in the present approach are very useful in uncovering the strong interactions, as for massive gravity [8].

It is convenient to make a $4+1$ split and write the action in terms of ADM-like variables [11]: the lapse $N = (G^{55})^{-1/2}$, the shift $N_\mu = G_{5\mu}$, and the 4D metric $\gamma_{\mu\nu} = G_{\mu\nu}$ on surfaces of constant $y = x^5$:

$$S_{\text{bulk}} = 2M_5^3 \int d^4x \int_0^\infty dy \sqrt{-\gamma} N \left[R(\gamma) - K^{\mu\nu}K_{\mu\nu} + K^2\right], \tag{3.2}$$

where

$$K_{\mu\nu} = \frac{1}{2N}(\dot{\gamma}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu) \tag{3.3}$$

is the extrinsic curvature. Here, 4D indices are raised and lowered with $\gamma_{\mu\nu}$, D_μ is the covariant derivative with respect to the 4D metric $\gamma_{\mu\nu}$, and the dot denotes a derivative with respect to y. Note that only first derivatives appear in the action and there is no boundary (Gibbons-Hawking) term in this formulation [11, 12] (see also [13]).

In order to integrate out the bulk fields we must choose a gauge for them. We want to choose a gauge such that the propagator has manifestly good high-energy behavior, so we choose de Donder gauge. We write

$$G_{MN} = \eta_{MN} + H_{MN}\,, \qquad \gamma_{\mu\nu} = \eta_{\mu\nu} + \zeta_{\mu\nu}\,, \tag{3.4}$$

and add the gauge fixing term

$$\mathcal{L}_{\text{bulk,gf}} = -M_5^3 F^M F_M\,, \tag{3.5}$$

where

$$F_M = \partial^N H_{MN} - \frac{1}{2}\partial_M H\,. \tag{3.6}$$

Classically, this imposes the gauge $F_M = 0$. In terms of the $4+1$ split,

$$\mathcal{L}_{\text{bulk,gf}} = -M_5^3 \left[\left(\partial^\mu \zeta_{\mu\nu} - \frac{1}{2}\partial_\nu \zeta - \frac{1}{2}\partial_\nu H_{55} + \dot{N}_\nu\right)^2 + \left(\partial^\mu N_\mu + \frac{1}{2}\dot{H}_{55} - \frac{1}{2}\dot{\zeta}\right)\right]^2. \tag{3.7}$$

This leaves residual gauge freedom under infinitesimal transformations satisfying

$$\Box_5 \Xi_M = 0\,. \tag{3.8}$$

Because we require Ξ_M to be well-behaved at infinity, and $\Xi_5| = 0$, we see that Ξ_5 is completely fixed but there is a residual gauge freedom, parameterized by $\xi_\mu = \Xi_\mu|$. Explicitly, the residual gauge freedom is

$$\Xi_\mu = e^{-y\Delta}\xi_\mu\,. \tag{3.9}$$

This residual gauge freedom acts on the boundary fields at linear order as

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,, \qquad \delta N_\mu = -\Delta \xi_\mu\,, \qquad \delta h_{55} = 0\,. \tag{3.10}$$

We now integrate out the bulk fields to obtain the quadratic boundary action. We solve the bulk equations of motion with boundary conditions

$$\bar{H}_{MN} = h_{MN}\,. \tag{3.11}$$

The equations of motion in de Donder gauge are

$$\Box_5 \left(\bar{H}_{MN} - \frac{1}{2}\eta_{MN}\bar{H} \right) = 0\,, \tag{3.12}$$

with solution

$$\bar{H}_{MN} = e^{-y\Delta} h_{MN}\,. \tag{3.13}$$

The induced boundary action is

$$\begin{aligned}\Gamma = M_5^3 \int d^4x \Bigg[&-\frac{1}{2} h^{\mu\nu}\Delta h_{\mu\nu} + \frac{1}{4} h_4 \Delta h_4 + \frac{1}{2} h_4 \Delta h_{55} - \frac{1}{4} h_{55}\Delta h_{55} - \\ &- N^\mu \Delta N_\mu - N^\mu (\partial_\mu h_4 + \partial_\mu h_{55} - 2\partial^\nu h_{\mu\nu}) \Bigg]\,. \end{aligned} \tag{3.14}$$

This is invariant under the gauge transformations eq. (3.10). In fact, in terms of the invariant combination

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{\Delta}(\partial_\mu N_\nu + \partial_\nu N_\mu) = -\frac{1}{\Delta} K_{\mu\nu} \tag{3.15}$$

we have

$$\Gamma = M_5^3 \int d^4x \left[-\frac{1}{2}\tilde{h}^{\mu\nu}\Delta \tilde{h}_{\mu\nu} + \frac{1}{4}\tilde{h}_4 \Delta \tilde{h}_4 + \frac{1}{2}\tilde{h}_4 \Delta h_{55} - \frac{1}{4} h_{55}\Delta h_{55} \right]. \tag{3.16}$$

The induced boundary action must be added to the DGP kinetic term on the boundary:

$$\mathcal{L}_{\text{bdy,DGP}} = M_4^2 \left[-\frac{1}{2}(\partial_\mu h_{\nu\rho})^2 + (\partial^\mu h_{\mu\nu})^2 - \partial^\mu h_4 \partial^\nu h_{\mu\nu} + \frac{1}{2}(\partial_\mu h_4)^2 \right]. \tag{3.17}$$

We fix the remaining gauge freedom parameterized by ξ_μ by adding a gauge fixing term

$$\mathcal{L}_{\text{bdy,gf}} = -M_4^2 \left(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h_4 + m N_\nu \right)^2, \tag{3.18}$$

where $m = M_5^3/M_4^2$ is the DGP scale. This gauge fixing makes the large DGP kinetic term invertible, and also eliminates the mixing between $h_{\mu\nu}$ and N_μ. The complete quadratic boundary lagrangian is then

$$\begin{aligned}\mathcal{L}_{\text{bdy}} = M_4^2 \Bigg[&\frac{1}{2} h^{\mu\nu}(\Box_4 - m\Delta) h_{\mu\nu} - \frac{1}{2} h_4 (\Box_4 - m\Delta) h_4 - \\ &- m N^\mu (\Delta + m) N_\mu + \frac{1}{2} m h_4 \Delta h_{55} - m N^\mu \partial_\mu h_{55} - m\frac{1}{4} h_{55}\Delta h_{55} \Bigg]\,. \end{aligned} \tag{3.19}$$

To see the strongly interacting mode, we consider the scalar modes

$$h_{\mu\nu} = \phi\eta_{\mu\nu}\,, \qquad N_\mu = \frac{1}{\Delta}\partial_\mu \sigma\,, \tag{3.20}$$

and h_{55}. In the regime $p \gg m$ the leading terms are

$$\mathcal{L}_{\rm bdy} \simeq M_4^2 \left[-2\phi\Box\phi + 2m\phi\Delta h_{55} - m\left(\sigma + \frac{1}{2}h_{55}\right)\Delta\left(\sigma + \frac{1}{2}h_{55}\right)\right]. \tag{3.21}$$

From this we see that there is one scalar mode that gets a kinetic term only through mixing with $h_{\mu\nu}$. This mode can be parameterized by

$$N_\mu = \partial_\mu \pi\,, \qquad h_{55} = -2\Delta\pi\,. \tag{3.22}$$

We can diagonalize the full kinetic term by defining

$$N'_\mu = N_\mu - \partial_\mu\pi\,, \qquad h'_{\mu\nu} = h_{\mu\nu} + m\pi\eta_{\mu\nu}\,, \tag{3.23}$$

and we obtain

$$\mathcal{L}_{\rm bdy} \simeq M_4^2 \left[\frac{1}{2}h'_{\mu\nu}\Box_4 h'_{\mu\nu} - \frac{1}{4}h'_4\Box_4 h'_4 - mN'^\mu\Delta N'_\mu + 3m^2\pi\Box_4\pi\right]. \tag{3.24}$$

The small coefficient of the π kinetic term is the origin of the strong interactions in this theory.

We can characterize the strongly-interacting mode in another way, which makes the generalization to curved backgrounds more transparent. The mode we found can be characterized by the following three properties: (i) it solves the linearized bulk equations of motion; (ii) $H_{\mu\nu} = 0$; (iii) it obeys the de Donder gauge-fixing condition

$$F_N = \partial^M H_{MN} - \frac{1}{2}\partial_N H = 0\,. \tag{3.25}$$

To see this, note that $H_{5\mu}$ and H_{55} can be locally gauged away, so any configuration satisfying these conditions must be pure gauge in the bulk at linear order. In fact, the mode eq. (3.22) extended into the bulk is

$$H_{\mu\nu} = 0\,, \qquad H_{5\mu} = \partial_\mu\Xi_5\,, \qquad H_{55} = 2\dot{\Xi}_5\,, \tag{3.26}$$

where

$$\Xi_5 = e^{-y\Delta}\pi\,. \tag{3.27}$$

Since $\Xi_5| \neq 0$ this gauge transformation is not a symmetry of the full action with boundary. The boundary shifts under this transformation, so this can be viewed as a 'brane bending mode.' This is the only nontrivial configuration with the three properties described above. Beyond linear order, $\Xi_5|$ and π have different interactions, since $\Xi_5|$ (unlike π) affects the 4D induced metric at order $(\Xi_5|)^2$

In fact, one could have anticipated by purely geometrical and physical arguments which mode, if any, could interact strongly. The strong mode should be related with the UV properties at the boundary, so we expect it to correspond to a trivial bulk geometry away from the brane. This is to say that the mode should be pure gauge in the bulk. Moreover, it should also not correspond to sizeable curvature of the induced boundary geometry: this is because the large DGP Einstein term disfavors intrinsic curvature. Since both the bulk and brane geometry should not be excited, the only remaining geometrical object that can be excited by the mode is the extrinsic curvature of the boundary, describing its shape as seen by a 5D observer. Up to trivial 4D reparametrizations, the only mode satisfying the above three requirements is precisely eq. (3.26). Notice that the second requirement bears similarity to the case of massive non-abelian gauge theory. There, the strongly interacting Goldstones are the the pure gauge configurations $A_\mu = U^\dagger \partial_\mu U$ for which the gauge kinetic term vanishes. The gauge kinetic term, whose coefficient $1/g^2$ in principle can be very large, is the analog of the DGP term.

We now turn to the question of higher-dimension operators in the effective theory. Bulk interactions with higher powers of H_{MN} will give rise to boundary interactions of the form

$$\Delta\mathcal{L}_{\rm bdy} \sim M_5^3 \partial (N_\mu)^p (\partial\pi)^q \sim m M_4^2 \partial \left(\frac{\hat{N}_\mu}{m^{1/2} M_4}\right)^p \left(\frac{\partial\hat{\pi}}{m M_4}\right)^q , \tag{3.28}$$

where

$$\hat{\pi} \sim m M_4 \pi \,, \qquad \hat{N}_\mu \sim m^{1/2} M_4 N_\mu \tag{3.29}$$

are the fields with unit kinetic term. From this we can read off a strong interaction scale

$$\Lambda_{(p,q)} \sim \left(m^{p/2+q-1} M_4^{p+q-2}\right)^{1/(3p/2+2q-3)} . \tag{3.30}$$

The lowest scale occurs for $p = 0$, $q = 3$ (cubic π interactions), which gives a scale

$$\Lambda \sim (m^2 M_4)^{1/3} . \tag{3.31}$$

Higher derivative terms in the bulk give rise to terms with additional powers of ∂/M_5. Since $M_5 \gg \Lambda$, these will give weaker interactions.

Notice that cubic terms cannot be canceled by changing the gauge condition. Suppose that we modify the gauge-fixing condition by adding non-linear terms in H_{MN}: $F'_M = F_M + \mathcal{O}(H^2)$. Since the mode eq. (3.22) obeys $F_M = 0$, the new gauge condition becomes $F'_M = O(\pi^2)$. This change produces only terms of order π^4 or higher in the boundary action.

We now show that the cubic terms are present by computing them explicitly. We must evaluate the cubic terms in π in the configuration (see eq. (3.22))

$$N_\mu = \partial_\mu \Pi \,, \qquad H_{55} = 2\partial_5 \Pi \,, \tag{3.32}$$

where

$$\Pi = e^{-y\Delta}\pi \,. \tag{3.33}$$

To find the cubic terms in the bulk action eq. (3.2) we need N to linear order and $K_{\mu\nu}$ to quadratic order:

$$\begin{aligned} N &= 1 + \frac{1}{2} H_{55} \\ K_{\mu\nu} &= \frac{1}{2}\left(1 - \frac{1}{2} H_{55}\right)(\partial_\mu N_\nu + \partial_\nu N_\mu) \,. \end{aligned} \tag{3.34}$$

The cubic terms involving π and N_μ are

$$\Delta\mathcal{L}_{\rm bulk} = M_5^3\left[\frac{1}{4}H_{55}(\partial_\mu N_\nu - \partial_\nu N_\mu)^2 + \partial^\mu H_{55}N_\mu\partial^\nu N_\nu - \partial^\nu H_{55}N_\mu\partial^\mu N_\nu\right]. \tag{3.35}$$

Using eq. (3.32) and $\Delta\Pi = -\dot{\Pi}$, this can be written

$$\begin{aligned}\Delta\mathcal{L}_{\rm bulk} &= 2M_5^3(\partial_\mu\Delta\Pi)(\partial^\mu\Pi\Box_4\Pi - \partial_\nu\Pi\partial^\mu\partial^\nu\Pi)\\ &= -M_5^3\partial_y\left[\partial_\mu\Pi\partial^\mu\Pi\Box_4\Pi\right].\end{aligned} \tag{3.36}$$

Integrating this solution over y, we obtain

$$\Delta\mathcal{L}_{\rm bdy} = M_4^2 m\partial^\mu\pi\partial_\mu\pi\Box_4\pi\,. \tag{3.37}$$

To see that this cubic interaction represents a physical effect, we can compute the correlation function of three stress-energy tensors on the brane. For kinematics where all momenta are space-like off-shell with $p \gg m$, the cubic interaction computed above dominates the amplitude, which becomes strong at the scale eq. (3.31). Similarly, by studying the Feyman diagrams, one finds that eq. (3.37) leads to a non-trivial 4-point scattering amplitude that violates unitarity at the scale Λ.

When we include quantum corrections, we expect all operators consistent with symmetries to be generated, and we expect an infinite number of terms that get strong at the scale Λ. Indeed, for the subset of logarithmically divergent graphs, we must include the associated operators in order to be consistent with unitarity.[4] These terms must be localized at the boundary, since the cutoff for bulk interactions far from the boundary is $M_5 \gg \Lambda_{\rm DGP}$. These interactions must respect 4D Lorentz invariance and must be local when written in terms of the geometrical quantities $R_{\mu\nu\rho\sigma}(\gamma)$ and $K_{\mu\nu}$. In order to zoom in on the strong interactions it is convenient to take a limit where $\Lambda = M_5^2/M_4$ is fixed and $M_4, M_5 \to \infty$, so that also $m = M_5^3/M_4^2 \to 0$. This is the analogue of the $g \to 0$ with $f_\pi = m_V/g$ fixed limit of massive non-abelian gauge theory. In this limit, the geometrical objects reduce to their linearized approximation

$$\begin{aligned}m^{-2}R_{\mu\nu} &= \frac{\partial_\mu\partial_\nu\hat{\pi}}{\Lambda^3} + O\left(\frac{m^2\partial\hat{\pi}\partial\hat{\pi}}{\Lambda^6}\right),\\ m^{-1}K_{\mu\nu} &= \frac{\partial_\mu\partial_\nu\hat{\pi}}{\Lambda^3} + O\left(\frac{m^2\partial\hat{\pi}\partial\hat{\pi}}{\Lambda^6}\right)\end{aligned} \tag{3.38}$$

where $\hat{\pi} = \pi/M_4 m$ is the canonically normalized field. By these equations we expect the terms that get strong at the scale Λ to have the form

$$\Delta\mathcal{L}_{\rm bdy} \sim \Lambda^4\left(\frac{\partial}{\Lambda}\right)^n\left(\frac{R(\gamma)}{m^2}\right)^p\left(\frac{K}{m}\right)^q. \tag{3.39}$$

Note that the theory becomes strongly coupled whenever the 4D curvature is of order m^2. We will comment further on this point in section 4 below.

[4] In other words, logarithmic divergences correspond to the RG evolution, so that the coefficient of the corresponding operators cannot be set to zero at all scales. Power divergent effects are not calculable and could consistently be set to zero, for example by using dimensional regularization.

Note also that the cubic interaction eq. (3.37) is nonlocal when written in terms of geometrical quantities (since $K \sim \partial^2\pi$ and $R \sim \partial^2\pi$). This means that loops of π fields should not renormalize this interaction, and that the divergent part of loop diagrams involving this interaction should be expressible as a function of $\partial^2\pi$.[5] This non-renormalization theorem follows simply by integration by parts. Consider any 1PI diagram with an external line coming from one of the factors of π with only one derivative. Because the diagram is 1PI, both of the other π factors attach to internal lines. We then have

$$\partial^\mu\pi_{\rm ext}\partial_\mu\pi_{\rm int}\Box_4\pi_{\rm int} = \partial^\mu\pi_{\rm ext}\partial_\nu\left[\partial_\mu\pi_{\rm int}\partial_\nu\pi_{\rm int} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\pi_{\rm int}\partial_\rho\pi_{\rm int}\right], \tag{3.40}$$

which is a function of $\partial^2\pi_{\rm ext}$ after integration by parts. This gives a nice check of the consistency of this framework.

We now consider the possibility of adding local terms to cancel the cubic π interaction. This is indeed possible in the case of massive gravity [8], where it allows one to raise the scale of strong interactions from $\Lambda_5 = (m_G^4 M_{\rm P})^{1/5}$ to $\Lambda_3 = (m_G^2 M_{\rm P})^{1/3}$. We will show that this is not possible in the DGP model. Local interactions that start at cubic order in the fields have the form R^3, K^3, etc., but these are of order $\mathcal{O}(\partial^6\pi^3)$ rather than $\mathcal{O}(\partial^4\pi^3)$. To cancel the cubic term of of eq. (3.37), we must use terms of the form

$$\Delta\mathcal{L}_{\rm bdy} \sim \frac{M_4^2}{m^2}R^2 + \frac{M_4^2}{m}RK + M_4^2 K^2\,. \tag{3.41}$$

Because of kinetic mixing between π and h, these terms contain terms of order $\mathcal{O}(\partial^4\pi^3)$. However, it is easy to see that these terms modify the h propagator at the scale m, taking us outside the original DGP model. Indeed, they give rise to massless ghost resonances with width $\sim m$ in some channel. This is not surprising, since we are adding higher derivative terms. For example, in the case of R^2 terms, we can avoid $\mathcal{O}(\partial^4\pi^2)$ terms by using the contraction $R^{\mu\nu}R_{\mu\nu} - R^2$, but this does not avoid ghosts in the graviton channel. In the case of K^2 terms one can avoid the ghost in the graviton channel, since this is a two derivative interaction. However there will always be a ghost in the $J = 1$ channel, the N_μ field. This can also be seen by working out the full graviton propagator on the brane in the unitary gauge: the piece involving the $J = 1$ projector comes with the wrong sign. The addition of $M_4^2 K^2$ is fully analogous to the addition of $\frac{1}{g^2}A_\mu\Box A^\mu$ to a massive gauge theory: in the Stuckelberg gauge there is a four derivative term for the Goldstone boson.

4. Classical instabilities

In this section we study a classical solution to the DGP model in which the stress-energy tensor on the brane satisfies the dominant energy condition, yet the brane has negative energy from the 5D point of view. When the boundary has the topology of R^4 it is difficult to define the 5D energy, which is presumably infinite. We therefore look for static solution

[5] A full calculation should include loops of bulk fields as well. However, the scaling argument shows the leading interactions at the scale Λ are expressible in terms of the interactions of the π field.

where the spatial sections of the boundary have topology S^3, the bulk is 'outside' the S^3, and the solution is $O(4)$ symmetric. The geometry of the boundary therefore corresponds to a spatially compact static cosmological solution, similar to the Einstein universe.

By Birkhoff's theorem, the metric outside the boundary is the 5D Schwarzschild metric

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2 d\Omega_3^2\,, \qquad f = \sqrt{1 - \frac{R_S^2}{r^2}} \tag{4.1}$$

where R_S is the Schwarzschild radius. The boundary is at a fixed value of $r > 0$ in these coordinates. Because this solution asymptotes to 5D flat space at infinity, the energy (mass) of the solution is well-defined:

$$M = 32\pi M_5^3 R_S^2\,. \tag{4.2}$$

For $R_S^2 < 0$ such a solution has negative energy. It is a negative-mass Schwarzschild solution with the naked singularity cut out by the boundary.

The most general form of the stress-energy tensor on the brane compatible with the symmetries is

$$T_{00} = -\rho\gamma_{00}\,, \qquad T_{ij} = +p\gamma_{ij}\,, \tag{4.3}$$

where ρ is the energy density and p is the pressure. We impose the equation of state

$$p = w\rho\,. \tag{4.4}$$

We look for solutions satisfying the dominant energy condition, which requires

$$\rho \geq 0\,, \qquad -1 \leq w \leq 1\,. \tag{4.5}$$

The bulk Einstein equations are satisfied by the metric eq. (4.1). The only additional equation that must be satisfied is

$$4M_4^2 \mathcal{G}_{\mu\nu}(\gamma) - 4M_5^3(K_{\mu\nu} - \gamma_{\mu\nu}K) = T_{\mu\nu}\,, \tag{4.6}$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is a stress tensor on the boundary. Eq. (4.6) follows simply from the variation of the full action with free boundary conditions. (There is no junction equation in this approach since there is no 'other side' to the boundary.) In the metric eq. (4.1), we can use eq. (3.3) to obtain

$$K_{\mu\nu} = \frac{1}{2} f \partial_r \gamma_{\mu\nu}\,. \tag{4.7}$$

The boundary equation (4.6) then gives

$$4M_4^2 \frac{3}{r^2} - 4M_5^3 \frac{3f}{r} = \rho\,, \tag{4.8}$$

$$4M_4^2 \frac{1}{r^2} - 4M_5^3 \frac{1}{r}\left(f + \frac{1}{f}\right) = -w\rho\,. \tag{4.9}$$

Note that when $M_5 = 0$ the 4D solution reduces to the standard Einstein static universe with $w = -1/3$, $\rho = 12M_4^2/r^2$. When $M_4 = 0$, eq. (4.6) is equivalent to the usual Israel junction conditions, and eqs. (4.8) and (4.9) have no solutions satisfying the dominant energy condition. (In fact, they have no solutions even if the stress energy tensor is allowed to be written as a negative tension term plus a term satisfying the dominant energy condition.)

The constraint $\rho \geq 0$ gives

$$rf \leq \frac{1}{m} . \tag{4.10}$$

Combining eqs. (4.8) and (4.9) we obtain

$$w = -\frac{1}{3} + \frac{4M_5^3}{\rho r f} , \tag{4.11}$$

which shows that $w \geq -1$ is satisfied whenever $\rho \geq 0$. The condition $w \leq 1$ gives

$$rf + \frac{r}{4f} \leq \frac{1}{m} . \tag{4.12}$$

Since this is clearly more restrictive than eq. (4.10), this is the only condition that needs to be checked. For $M < 0$ ($R_S^2 < 0$), the left-hand side of eq. (4.12) approaches $|R_S|$ as $r \to 0$ and increases monotonically with r, so we obtain a solution for $|R_S| < 1/m$. This means that the energy cannot be made arbitrarily negative in this model (see eq. (4.2)).

Note that the minimal 4D curvature of a negative energy solution is $\mathcal{O}(m^2)$. For the critical zero energy solution $f \to 1$, and eq. (4.12) implies $r \lesssim 1/m$, so $\mathcal{G}_{\mu\nu} \gtrsim m^2$. However, eq. (3.39) shows that such curvature is also the critical curvature where the derivative expansion of the effective quantum field theory breaks down. So the negative energy solutions to lie at the edge of the regime of validity of our theory.

Another way of arguing the same point is the following. The instability appears for 4D energy density $\rho = \mathcal{O}(M_4^2 m^2)$. Significantly, this is also the energy density of a gravitational source for which the cubic interaction term in eq. (3.37) becomes comparable to the kinetic term of the scalar π, defined in section 3. To see this, recall that π couples to the stress-energy tensor with strength m (see eq. (3.23)). Then, to linear order in the source, $\Box_4 \pi \sim T^\mu_\mu / M_4^2 m$. Substituting this estimate into the cubic interaction term eq. (3.37) we have

$$\Delta \mathcal{L}_{\rm bdy} \sim T^\mu{}_\mu \partial^\nu \pi \partial_\nu \pi . \tag{4.13}$$

This term becomes of the same order as the π kinetic term when $T^\mu{}_\mu \sim M_4^2 m^2$. We conclude that the negative energy solutions appear only at the edge of validity of the effective theory, and the theory with cutoff of order $\Lambda \sim (m^2 M_4)^{1/3}$ is safe from instabilities.

5. Curved backgrounds

A noteworthy aspect of massive gravity is that when propagating on a curved background, it behaves very differently than in flat space. In AdS space, there is no vDVZ [3] discontinuity [9], while in dS a light massive graviton becomes a ghost [14]. These unusual features find a simple explanation when massive gravity is rendered covariant by adding a Goldstone vector [15, 8] A_μ. At linear order, A_μ appears in the combination $h_{\mu\nu} - \bar{D}_{(\mu} A_{\nu)}$.

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Here $\bar{D}_\mu$ is the covariant derivative of the 4D background. The difference with flat space originates from the fact that at nonzero cosmological constant Λ, the kinetic term of the strongly-interacting scalar mode π inside A_μ, $A_\mu = \bar{D}_\mu \pi$, receives a contribution proportional to $-\Lambda$. This contribution suppresses the cubic interactions of π when $\Lambda < 0$, and makes the graviton a ghost at small mass when $\Lambda > 0$ [8].

In DGP, we find an analogous phenomenon. We study the most general curved background in the absence of matter. The sources of curvature are given by bulk and brane cosmological constants and by gravity itself. In the presence of matter sources there would be extra model dependent modes that mix with the scalar and vector components of the metric. We first give a general argument, then consider two important special cases: one with a boundary with de Sitter geometry, the other a Randall-Sundrum model with a DGP kinetic term on the boundaries.

5.1 General discussion

We consider a general 5D background metric $\bar{G}_{MN}$ satisfying Einstein equations with cosmological constant

$$\bar{R}_{MN} - \frac{1}{2}\bar{G}_{MN}\bar{R} + \lambda_5 \bar{G}_{MN} = 0 \tag{5.1}$$

and study the linearized theory

$$G_{MN} = \bar{G}_{MN} + H_{MN}\,. \tag{5.2}$$

Here and in the following we contract indices using $\bar{G}_{MN}$, and we indicate covariant derivative, curvature, etc. associated with the background metric $\bar{G}_{MN}$ respectively by $\bar{\nabla}_M$, $\bar{R}_{MNRS}$ etc.. We generalize de Donder gauge to curved backgrounds by adding the gauge fixing term

$$\Delta\mathcal{L}_{\text{bulk,gf}} = -M_5^3 \bar{G}^{MN} F_M F_N\,, \tag{5.3}$$

where

$$F_M = \bar{\nabla}^N H_{MN} - \frac{1}{2}\bar{\nabla}_M H\,, \tag{5.4}$$

The linearized bulk equations of motion are then

$$\bar{\nabla}^2 \left(H_{MN} - \frac{1}{2}\bar{G}_{MN} H \right) + 2\bar{R}_{MRNS} H^{RS} = 0\,. \tag{5.5}$$

Following the discussion in the flat case, it is convenient to parameterize the modes of H_{MN} as follows. For simplicity, we will give the discussion using gaussian normal coordinates for the background metric: $\bar{G}_{5\mu} = 0$, $\bar{G}_{55} = 1$. Instead of $H_{5\mu}| = h_{5\mu}$ and $H_{55}| = h_{55}$, we use as boundary variables

$$\Xi_\mu| = \xi_\mu\,, \qquad \Xi_5| = \pi\,, \tag{5.6}$$

where Ξ_M satisfies the bulk equation

$$\bar{\nabla}^2 \Xi_M + \bar{R}_{MN}\Xi^N = 0\,. \tag{5.7}$$

This implies that the mode $H_{MN} = \bar{\nabla}_{(M}\Xi_{N)}$ satisfies the bulk equations of motion as well as the de Donder gauge fixing condition. We then parameterize a general bulk fluctuation as

$$H_{MN} = H^{(2)}_{MN} + \bar{\nabla}_{(M}\Xi_{N)} - \tilde{H}_{MN}\,, \tag{5.8}$$

where

$$\bar{\nabla}^2 H^{(2)}_{MN} = 0\,, \qquad \bar{\nabla}^2 \tilde{H}^{(2)}_{MN} = 0\,, \tag{5.9}$$

with boundary values

$$H^{(2)}_{\mu\nu}| = h_{\mu\nu}\,, \qquad H^{(2)}_{5\mu}| = 0\,, \qquad H^{(2)}_{55}| = 0\,, \tag{5.10}$$

and

$$\tilde{H}_{\mu\nu}| = \bar{\nabla}_{(\mu}\Xi_{\nu)}| = \bar{D}_{(\mu}\xi_{\nu)} + 2\bar{K}_{\mu\mu}\xi_5\,, \qquad \tilde{H}_{5\mu}| = 0\,, \qquad \tilde{H}_{55}| = 0\,, \tag{5.11}$$

where $\bar{D}_\mu$ is the covariant derivative with respect to the induced background metric $\bar{\gamma}_{\mu\nu}$. The $\tilde{H}_{MN}$ term in eq. (5.8) subtracts the contribution of the 'Goldstone' modes Ξ_M to the fluctuations of the induced boundary metric, which is then simply $H^{(2)}_{\mu\nu}| = h_{\mu\nu}$, the 'spin 2' mode. This ensures that ξ_μ and π as defined above do not appear in the large DGP kinetic term. We will see that in terms of these variables, the identification of the strong degrees of freedom is more direct. (Indeed, the combination N'_μ that diagonalizes the kinetic term in eq. (3.23) in the flat case is precisely the 'Goldstone' ξ_μ.)

The dependence on ξ_μ can be obtained simply by noting that under

$$h_{\mu\nu} \mapsto \bar{D}_{(\mu}\lambda_{\nu)}\,, \qquad \xi_\mu \mapsto \xi_\mu + \lambda_\mu\,, \qquad \pi \mapsto \pi\,, \tag{5.12}$$

the bulk fluctuation eq. (5.8) changes precisely by a residual gauge transformation Λ_M, satisfying $\Lambda_\mu| = \lambda_\mu$, $\Lambda_5 = 0$. eq. (5.12) is therefore a symmetry of the quadratic boundary action. Therefore, we can work out the action at $\xi_\mu = 0$ and restore the dependence on ξ_μ by the substitution

$$h_{\mu\nu} \to h_{\mu\nu} - \bar{D}_{(\mu}\xi_{\nu)}\,. \tag{5.13}$$

We now compute the boundary action for the modes above. For this we will need

$$\bar{\nabla}_5 H^{(2)}_{\mu\nu}| \simeq -\bar{\Delta} h_{\mu\nu}\,, \qquad \bar{\nabla}_5 \Xi_\mu| \simeq -\bar{\Delta}\xi_\mu\,, \qquad \bar{\nabla}_5 \Pi| \simeq -\bar{\Delta}\pi\,, \tag{5.14}$$

valid for modes with 4D wavelengths smaller than the scale of curvature, where

$$\bar{\Delta} = \sqrt{-\bar{D}^2}\,. \tag{5.15}$$

Eq. (5.14) follows from the fact that for small-wavelength fluctuations the curvature is irrelevant, so the result must reduce smoothly to the flat case. We will see how this arises in an explicit calculation in the next subsection. The contribution to the boundary effective action from the bulk Einstein action is then

$$\begin{aligned}\Delta\mathcal{L}_{\rm bdy,E} &= -M_5^3 h^{\mu\nu}\left[\sqrt{-\gamma}N(K_{\mu\nu} - \gamma_{\mu\nu}K)\right]_{\rm linearized} \\ &= -\sqrt{-\bar{\gamma}}M_5^3\left[\frac{1}{2}h^{\mu\nu}\bar{\Delta}h_{\mu\nu} - \frac{1}{2}h\bar{\Delta}h + h^{\mu\nu}\bar{D}_\mu\bar{D}_\nu\pi - h\bar{D}^2\pi + \cdots\right],\end{aligned} \tag{5.16}$$

where we have omitted terms of order $\bar{K}h^2$ and $\bar{K}h\bar{\Delta}\pi$ that are subleading at small 4D wavelengths. The contribution from the bulk de Donder gauge fixing term is

$$\begin{aligned}\Delta\mathcal{L}_{\rm bdy,bulk\,gf} &= -M_5^3\sqrt{-\bar{\gamma}}\left(H_{5M}F^M - \frac{1}{2}HF^5\right)\\ &= M_5^3\sqrt{-\bar{\gamma}}\Bigg[-\frac{1}{4}h\bar{\Delta}h - \pi(\bar{D}^\mu\bar{D}^\nu h_{\mu\nu} - \bar{D}^2h) + \\ &\qquad + 2\pi(\bar{K}_{\mu\nu}\bar{D}^\mu\bar{D}^\nu - \bar{K}\bar{D}^2)\pi + \cdots\Bigg]\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}F_\mu &= \bar{D}^\nu(h_{\mu\nu} - 2\bar{K}_{\mu\nu}\pi) - \frac{1}{2}\bar{D}_\mu(h - 2\bar{K}\pi) + \mathcal{O}(\bar{K}^2\pi, \bar{K}h),\\ F_5 &= \frac{1}{2}\bar{\Delta}(h - 2\bar{K}\pi) + \mathcal{O}(\bar{K}^2\pi, \bar{K}h),\end{aligned} \tag{5.18}$$

and we again omit terms that are subleading at small 4D wavelengths. At this point, it is trivial to include the mode ξ_μ by the substitution eq. (5.13). Note that π mixes with $h_{\mu\nu}$ only via the linearized Ricci scalar, so there is no mixing between π and ξ_μ, generalizing the result of flat case.

The full quadratic boundary action is therefore the sum of eqs. (5.16) and (5.17). In the small wavelength approximation, the only relevant change comes from the last two terms in eq. (5.17). To understand what they imply, we consider for simplicity a maximally symmetric background, for which

$$\bar{K}_{\mu\nu} = C\bar{\gamma}_{\mu\nu}\,. \tag{5.19}$$

Adding the DGP kinetic term and diagonalizing the kinetic term for π and $h_{\mu\nu}$ we obtain the π kinetic term

$$\mathcal{L}_{\rm kin} = 3M_4^2 m(m - 2C)\pi\bar{D}^2\pi\,. \tag{5.20}$$

Note that the π mode becomes a ghost for $C = \frac{1}{2}m$. We will see below that $C > 0$ for de Sitter space (see eq. (5.25)). Therefore, as in massive gravity, positive curvature increases the strength of the strongest interactions, while negative curvature decreases it.

One of the motivations for considering the DGP model is that it provides a source of 'dark energy' in the absence of 4D vacuum energy [2]. In this case, we can compute the relation between the constant C introduced above and the positive curvature of the present-day universe. From eq. (4.6), we have

$$4M_4^2\mathcal{G}_{\mu\nu}(\gamma) = T_{\mu\nu} - 12M_5^3C\gamma_{\mu\nu} \simeq T_{\mu\nu} - 4M_4^2\lambda_{\rm now}\gamma_{\mu\nu}\,. \tag{5.21}$$

This shows that C is positive, hence π is a ghost, for sufficiently small m. In the next subsection, we will see that in a DGP model with no cosmological constant in the bulk, π is a ghost in the regime where 4D vacuum energy does not contribute to the 4D curvature. More generally, we expect that positive 4D curvature decreases the strength of the π kinetic term, so that it makes the interactions of π even stronger than on a Minkowsky 4D background. Conversely, a negative 4D curvature weakens the π self-interactions, as in massive gravity. When m is much smaller than the curvature $|C|$, it even eliminates the vDVZ discontinuity already at linear order [9].

5.2 Explicit calculation: de Sitter space

We now consider the important special case of 4D de Sitter space in a DGP model with vanishing bulk cosmological constant. The solution has very simple 5D geometry: the bulk is flat, and the boundary is at

$$\eta_{\mu\nu} x^\mu x^\nu + y^2 = L^2 \tag{5.22}$$

in cartesian coordinates. It is more convenient to use coordinates where the background metric is

$$ds^2 = dr^2 - r^2 d\tau^2 + r^2 \cosh^2\tau \, d\Omega_3^2 \,. \tag{5.23}$$

The boundary is at $r = L$ in these coordinates. The boundary equation is

$$4M_4^2 \mathcal{G}_{\mu\nu}(\bar\gamma) - 4M_5^2 \left[\bar K_{\mu\nu} - \bar\gamma_{\mu\nu} \bar K \right] = -V_0 \bar\gamma_{\mu\nu} \,, \tag{5.24}$$

where $V_0 > 0$ is the vacuum energy on the boundary. It is straightforward to work out

$$\mathcal{G}_{\mu\nu}(\bar\gamma) = -\frac{3}{r^2}\bar\gamma_{\mu\nu} \,, \qquad \bar K_{\mu\nu} = +\frac{1}{r}\bar\gamma_{\mu\nu} \,. \tag{5.25}$$

This gives the relation between the 4D vacuum energy and the curvature:

$$\frac{V_0}{12 M_4^2} L^2 + mL = 1 \,. \tag{5.26}$$

For $V_0/M_4^2 \gg m^2$, this gives the usual relation between vacuum energy and curvature, but for $V_0/M_4^2 \ll m^2$ we get $L \simeq 1/m$ [2]. It is interesting that in this regime the 4D curvature is independent of the vacuum energy, but from eq. (5.20) we see that the Goldstone is always a ghost in this regime.

We now consider the explicit calculation of the boundary action. To understand the strong interactions, it is sufficient to consider the scalar modes

$$H_{MN} = \bar G_{MN} \Phi + (\bar n_M \bar\nabla_N \Sigma + \bar n_N \bar\nabla_M \Sigma) + \bar n_M \bar n_N \Omega \,, \tag{5.27}$$

where $\bar n_M$ is the normal vector $\bar n^5 = \bar n_5 = 1$, $\bar n^\mu, \bar n_\mu = 0$. A useful relation is

$$\bar\nabla_M \bar n_N = \frac{1}{r} \bar\gamma_{MN} \,, \tag{5.28}$$

where

$$\bar\gamma_{MN} = \bar G_{MN} - \bar n_M \bar n_N \tag{5.29}$$

is the induced metric on surfaces of constant r.[6] The de Donder equations of motion for the scalar modes defined above are

$$\begin{aligned}
\bar\gamma_{\mu\nu}\left(\bar\nabla^2 \Phi + \frac{2}{r^2}\Omega \right) + \frac{4}{r}\bar\nabla_\mu \bar\nabla_\nu \Sigma &= 0 \,, \\
\bar\nabla_\mu \left[\bar\nabla^2 \Sigma + \frac{2}{r}\bar\nabla_r \Sigma - \frac{4}{r^2}\Sigma + \frac{2}{r}\Omega \right] &= 0 \,, \\
\bar\nabla^2 \Omega + \bar\nabla^2 \Phi + 2\bar\nabla_r \left(\bar\nabla^2 \Sigma - \frac{4}{r^2}\Sigma \right) - \frac{8}{r^2}\Omega &= 0 \,.
\end{aligned} \tag{5.30}$$

[6] We can think of r as a bulk scalar. Geometrically, it is the proper distance of any point in the bulk to the boundary.

In our coordinates

$$\bar{\nabla}^2\Phi = \left[\partial_r^2 + \frac{4}{r}\partial_r + \frac{1}{r^2}\hat{\Box}_4\right]\Phi\,, \tag{5.31}$$

where $\hat{\Box}_4$ is the laplacian on S^3. Since $\hat{\Box}_4$ is independent of r, we can treat it as a parameter when solving the equations. (More formally, we could expand in eigenstates of $\hat{\Box}_4$.)

The behavior of the solutions is very easy to understand once we notice that they are homogeneous in r. The solutions therefore have the form

$$\Phi = \left(\frac{r}{L}\right)^A \phi\,, \qquad \Sigma = \left(\frac{r}{L}\right)^{A+1} \sigma\,, \qquad \Omega\left(\frac{r}{L}\right)^A \omega\,, \tag{5.32}$$

where A depends on $\hat{\Box}_4$. The solution is very simple for fluctuations with $\Box_4 \gg 1/L^2$ on the boundary. (Recall that L is the size of the 4D universe!) For these fluctuations $\hat{\Box}_4 \gg 1$ and the leading terms in the equations that determine A are simply

$$A^2 + \hat{\Box}_4 = 0\,, \tag{5.33}$$

with solution $A = \pm\sqrt{-\hat{\Box}_4}$. Good behavior at infinity (away from the boundary) requires the negative solution. Since $\hat{\Box}_4$ is the only expansion parameter, the corrections are

$$A = -\sqrt{-\hat{\Box}_4}\left[1 + \mathcal{O}\left(\frac{1}{\hat{\Box}_4}\right)\right], \tag{5.34}$$

and so on the boundary

$$A| = -L\bar{\Delta} + \mathcal{O}\left(\frac{1}{L\bar{\Delta}}\right). \tag{5.35}$$

Note that the corrections to A are smaller than the $1/L$ curvature corrections described in the previous subsection.

The conclusions depend only on the fact that the equations are second order and homogeneous in r, and so hold for general polarization states. This shows that

$$\partial_r H_{MN}| = -\bar{\Delta} H_{MN} + \mathcal{O}\left(\frac{H}{L^3\bar{\Delta}}\right). \tag{5.36}$$

The remainder of the calculation follows the previous subsection line by line.

5.3 DGP and Randall-Sundrum

It is instructive to apply the results of the previous section to the Randall-Sundrum (RS) model [10] with a DGP boundary term added. At the boundaries, we have

$$\bar{K}_{\mu\nu} = \pm L\bar{\gamma}_{\mu\nu}\,, \tag{5.37}$$

where $L = 1/k$ is the bulk AdS curvature length, and the $+$ $(-)$ sign corresponds to the IR (Planck) brane.

We first consider adding a DGP kinetic term on the Planck brane. As long as $k \gtrsim m$, eq. (5.20) shows that the scale of strong interactions is $(M_5 k)^{1/2} \gtrsim k$. We conclude that the DGP kinetic term does not lead to new strong interactions within the original regime of validity of the RS model. This is consistent with the holographic interpretation of the model as a 4D conformal field theory (CFT) coupled to gravity. The DGP kinetic term simply corresponds to a large coefficient for the gravity kinetic term in the UV, large enough to dominate the induced contribution to the Planck scale from the CFT. Note that the extrinsic curvature term in the boundary effective action is crucial for obtaining this result.

We now consider adding a DGP kinetic term to the IR (or 'TeV') brane. In this case, eq. (5.20) tells us that the theory has a ghost for $k > \frac{1}{2}m$. In fact, this instability comes from the radion itself becoming a ghost. To see this in a simple way, we consider the limit where the Planck brane is pushed to the boundary of AdS and the IR brane is at fixed position $y = 0$. In this limit the 4D zero mode graviton is decoupled from the physics on the IR brane. The only zero mode is the radion ϕ, which can be parameterized by [16]

$$ds^2 = e^{2ky}\left[1 + 2\phi(x)e^{-2ky}\right] dx_\mu dx^\mu + \left[1 - 2\phi(x)e^{-2ky}\right]^2 dy^2 . \tag{5.38}$$

Its kinetic term is

$$\mathcal{L}_{\rm kin} = 6\left(\frac{M_5^3}{k} - 2M_4^2\right)\phi\Box_4\phi\,, \tag{5.39}$$

in agreement with eq. (5.20).

This is interesting because it prevents a geometrical construction of a model with an isolated massive spin 2 particle. This model has only massive spin 2 KK modes, and when $M_4^2 \gg M_5^3/k$ one finds that the lightest spin 2 mode has an anomalously small mass of order $(km)^{1/2}$, while the remaining massive KK modes have mass of order k. If it were not for the instability, discussed above, for $m \ll k$ this model would reduce to an effective theory of a single massive graviton (and a radion) below the scale k.

6. Conclusions

We now summarize our results. First, we showed that the DGP model has strong interactions at distances shorter than $\lambda_3 \sim (\lambda_{\rm DGP}^2/M_4)^{1/3}$. The strong interactions are due to a scalar 'Goldstone' mode that obtains a kinetic term only by mixing with the transverse graviton polarizations, similar to massive gravity. The longitudinal Goldstone has a geometrical interpretation as a brane bending mode that keeps the induced metric on the brane fixed. Second, we showed that there are classical instabilities in the DGP model in the form of negative energy solutions. These solutions are at the edge of validity of the effective field theory with UV cutoff at the scale λ_3, giving further support to the conclusion that new physics is required at this scale. Finally, we considered the strong interactions in the presence of 4D and/or 5D curvature. We showed that positive (de Sitter) sign curvature makes the model more strongly interacting, and makes the strong mode a ghost for sufficiently large curvature. We also investigated the effect of a DGP kinetic term in the Randall-Sundrum model.

We conclude with some comments on the solution of the vDVZ [3] discontinuity problem suggested in ref. [5]. There, it was shown that around a classical source of Schwarzschild radius $R_{\rm S}$, a careful resummation of non-linear effects restores the phenomenologically correct Schwarzschild solution below a distance $R_* \sim (R_{\rm S}/m^2)^{1/3}$. Now, the curvature of

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the Schwarzschild metric is of order R_S/r^3, so that at the distance R_* the curvature is of order m^2. From inspection of eq. (3.39) we find that this is the critical curvature at which the quantum effective field theory breaks down. Stated otherwise, at the scale R_*, the effective quantum expansion parameter $\epsilon = \partial^2\hat{\pi}/\Lambda^3$ becomes order 1. Therefore, any statement about the behavior of the field at distances smaller than R_* requires knowledge (or assumptions) about the UV completion of the π sector.

One possibility is to assume that when $\epsilon \gg 1$ the infinite series of counterterms saturates and the quantum correction to the effective action stays of the order of the result at $\epsilon \sim 1$. In this case, the contribution from the counterterms is suppressed compared to the tree-level contribution (which has fewer derivatives) by $\sim 1/(M_4 R_S)^2$. In this scenario, there is a range of scales where the effect of ref. [5] works. In this case, one still has to find a way to avoid the instabilities associated with curvature of order m^2, found above. Moreover, this does not rescue the model phenomenologically, since at least at the length scale $\lambda_3 \sim 1000\,\mathrm{km}$, gravity becomes sensitive to the details of the UV completion.

Note added: ref. [17], which appeared after our paper was completed, also investigates strong interactions in the DGP model. The approach of that work includes some but not all the effects to the order considered.

Acknowledgments

We would like to thank Raman Sundrum for collaboration in the early stages of this project. We also thank Nima Arkani-Hamed, Thibault Damour, Sergei Dubovsky, Gia Dvali, Gregory Gabadadze, Rob Leigh and Valery Rubakov for useful and stimulating discussions. M.A.L. was supported by NSF grant PHY-0099544. M.P. conducted a significant part of research related to this paper while on sabbatical at the Institute for Advanced Studies, Princeton NJ, and the Scuola Normale Superiore, Pisa, Italy. He would like to thank their gracious hospitality and support. M.P. is supported in part by NSF grants PHY-0070787 and PHY-0245068. R.R. would like to thank the Scuola Normale Superiore for hospitality during this project.

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